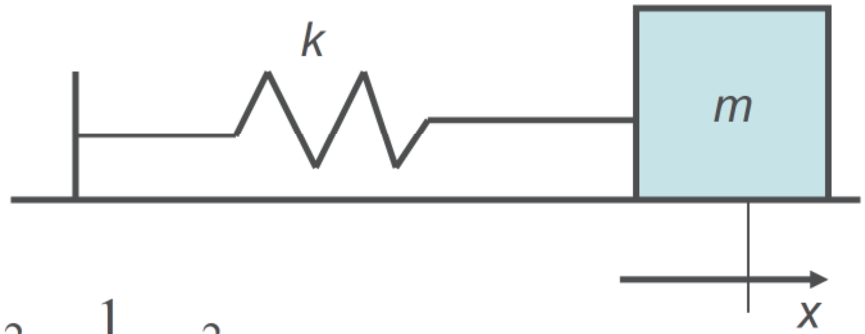
Example of Linear Spring Mass System and Frictionless Table: **The Steps**



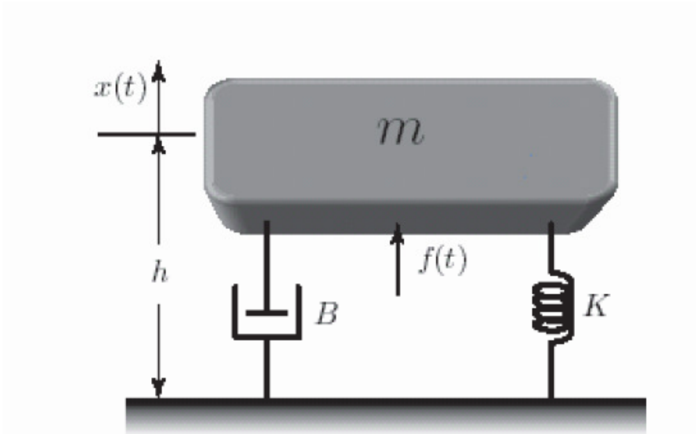
$$\text{Lagrangian}: L = K_e - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\text{Lagrang's Equation}: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\text{Do the derivatives}: \frac{\partial L}{\partial \dot{q}_i} = m\dot{x};\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = m\ddot{x};\ \frac{\partial L}{\partial q_i} = -kx$$

$$\text{Combine all together}: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = m\ddot{x} + kx = 0$$

Mechanical Example: Mass-Spring Damper



$$Ke = \frac{1}{2} m\dot{x}^2$$

$$V = \frac{1}{2} Kx^2 + mg(h+x)$$

$$L = K_e - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} Kx^2 - mg(h+x)$$

$$P = \frac{1}{2} B\dot{x}^2$$

We have the generalized coordinate $q = x$, and thus with the applied force $Q = f$, we write the Lagrange equation :

$$f = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial P}{\partial \dot{x}}$$

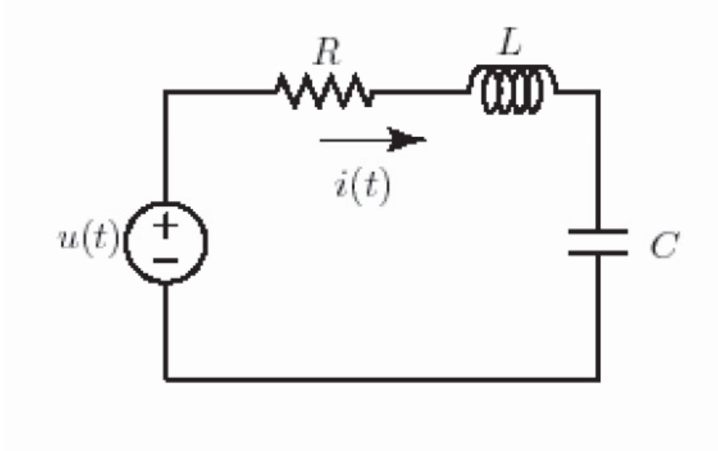
$$= \frac{d}{dt}(\frac{\partial}{\partial \dot{x}}(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} Kx^2 - mg(h+x)))$$

$$- \frac{\partial}{\partial x}(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} Kx^2 - mg(h+x)) + \frac{\partial}{\partial \dot{x}}(\frac{1}{2} B\dot{x}^2)$$

$$= \frac{d}{dt}(m\dot{x}^2) - (-Kx - mg) + (B\dot{x})$$

$$= m\ddot{x} + Kx + mg + B\dot{x}$$

Electrical Example: *RLC* Circuit



$$K_e = \frac{1}{2} L\dot{q}^2$$

$$V = \frac{1}{2C} q^2$$

$$L = K_e - V = \frac{1}{2} L\dot{q}^2 - \frac{1}{2C} q^2$$

$$P = \frac{1}{2} R\dot{q}^2$$

We have the generalized coordinate q (charge), and with the applied force $Q = u$, we have

$$u = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial P}{\partial \dot{q}}$$

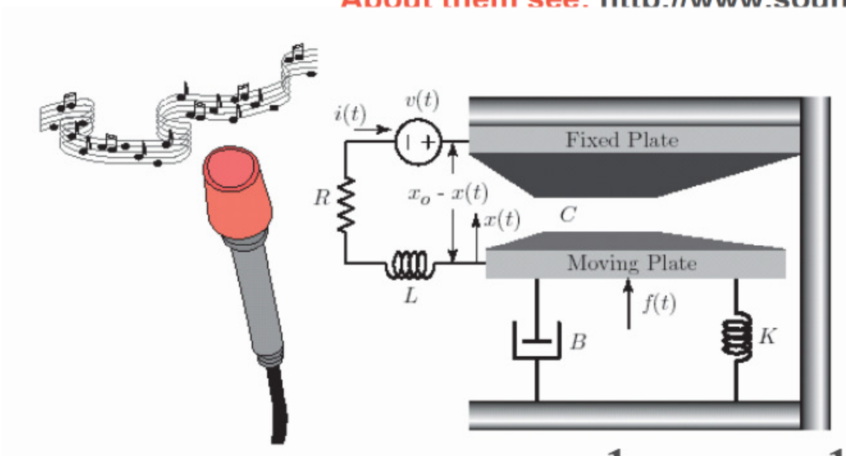
$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2} L\dot{q}^2 - \frac{1}{2C} q^2\right)\right) - \frac{\partial}{\partial q}\left(\frac{1}{2} L\dot{q}^2 - \frac{1}{2C} q^2\right) + \frac{\partial}{\partial \dot{q}}\left(\frac{1}{2} R\dot{q}^2\right)$$

$$= \frac{d}{dt}(L\dot{q}) + \frac{Q}{C} + R\dot{q} = L\ddot{q} + \frac{Q}{C} + R\dot{q} = L\frac{di}{dt} + v_c + Ri$$

$i = \dot{q}$ and $q = Cv_c$ for a capacitor. This is just KVL equation

Electromechanical System: Capacitor Microphone

About them see: http://www.soundonsound.com/sos/feb98/articles/capacitor.html



This system has two degrees of freedom (electrical and mechanical : charge q and displacement x from equilibrium)

$$K_e = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}m\dot{x}^2;\; V = \frac{1}{2C}q^2 + \frac{1}{2}Kx^2$$

$$C = \frac{\varepsilon A}{x_o - x}\left(\begin{array}{l}\varepsilon \text{ is the dielectric constant of the air (F/m),}\\ A \text{ is the area of the plate, } x_o - x \text{ is the plate separation}\end{array}\right)$$

$$V = \frac{1}{2\,\varepsilon\,A}(x_o - x)q^2 + \frac{1}{2}Kx^2;\; P = \frac{1}{2}R\dot{q}^2 + \frac{1}{2}B\dot{x}^2$$

$$L = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2\varepsilon A}(x_o - x)q^2 - \frac{1}{2}Kx^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}; \ \frac{\partial L}{\partial x} = \frac{q^2}{2\varepsilon A} - Kx; \frac{\partial \mathrm{P}}{\partial \dot{x}} = B\dot{x}$$

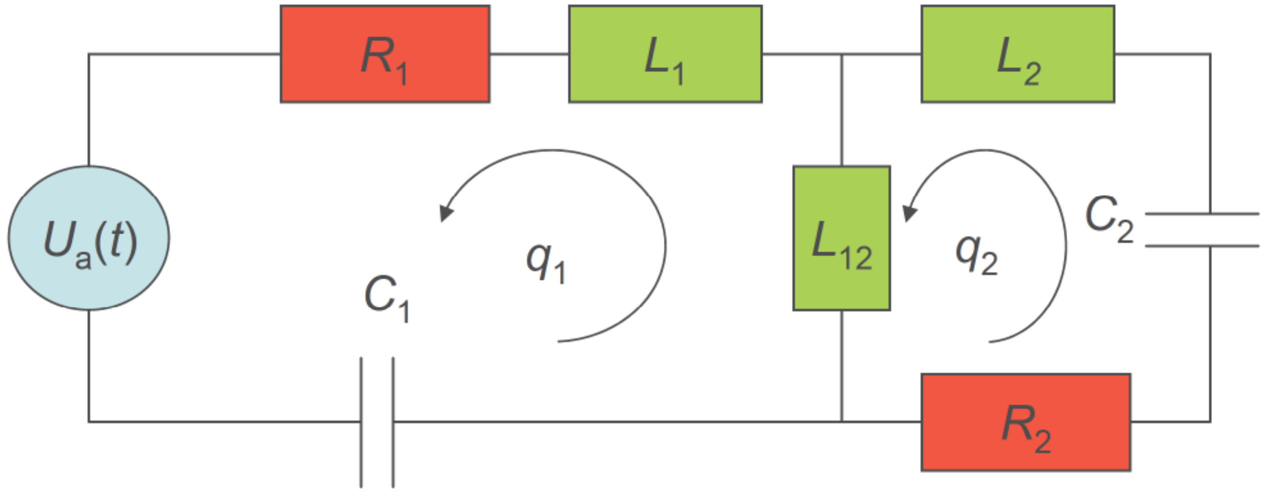
$$\frac{\partial L}{\partial \dot{q}} = L\dot{q}; \ \frac{\partial L}{\partial q} = \frac{(x_o - x)q}{\varepsilon A}; \frac{\partial P}{\partial \dot{q}} = R\dot{q}$$

Then we obtain the two Lagrange equations

$$m\ddot{x} + B\dot{x} + Kx - \frac{q2}{2\,\varepsilon\,A} = f$$

$$L\ddot{q} + R\dot{q} + \frac{1}{\varepsilon\,A}(x_o - x)q = v$$

Example: Two Mesh Electric Circuit



Assume q_1 and q_2 as the independent generalized coordinates, where q_1 is the electric charge in the first loop and q_2 is the electric charge in the second loop.

The generalized force applied to the system is denoted as Q_1

We should know that : $i_1 = \dot{q}_1;\ i_2 = \dot{q}_2;\ q_1 = \frac{i_1}{s};\ q_2 = \frac{i_2}{s};\ Q_1 = U_a(t).$

The total magnetic energy (kinetic energy) is :

$$K_e = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_{12}(\dot{q}_1 - \dot{q}_2)^2 + \frac{1}{2}L_2\dot{q}_2^2$$

$$\frac{\partial K_e}{\partial q_1} = 0; \ \frac{\partial K_e}{\partial \dot{q}_1} = (L_1 + L_{12})\dot{q}_1 - L_{12}\dot{q}_2$$

$$\frac{\partial K_e}{\partial q_2} = 0; \ \frac{\partial K_e}{\partial \dot{q}_2} = (L_2 + L_{12})\dot{q}_2 - L_{12}\dot{q}_1$$

Use the equation for the total electric energy (potential energy)

$$V = \frac{1}{2}\frac{q_1^2}{C_1} + \frac{1}{2}\frac{q_2^2}{C_2}; \ \frac{\partial V}{\partial q_1} = \frac{q_1}{C_1} \text{ and } \frac{\partial V}{\partial q_2} = \frac{q_2}{C_2}$$

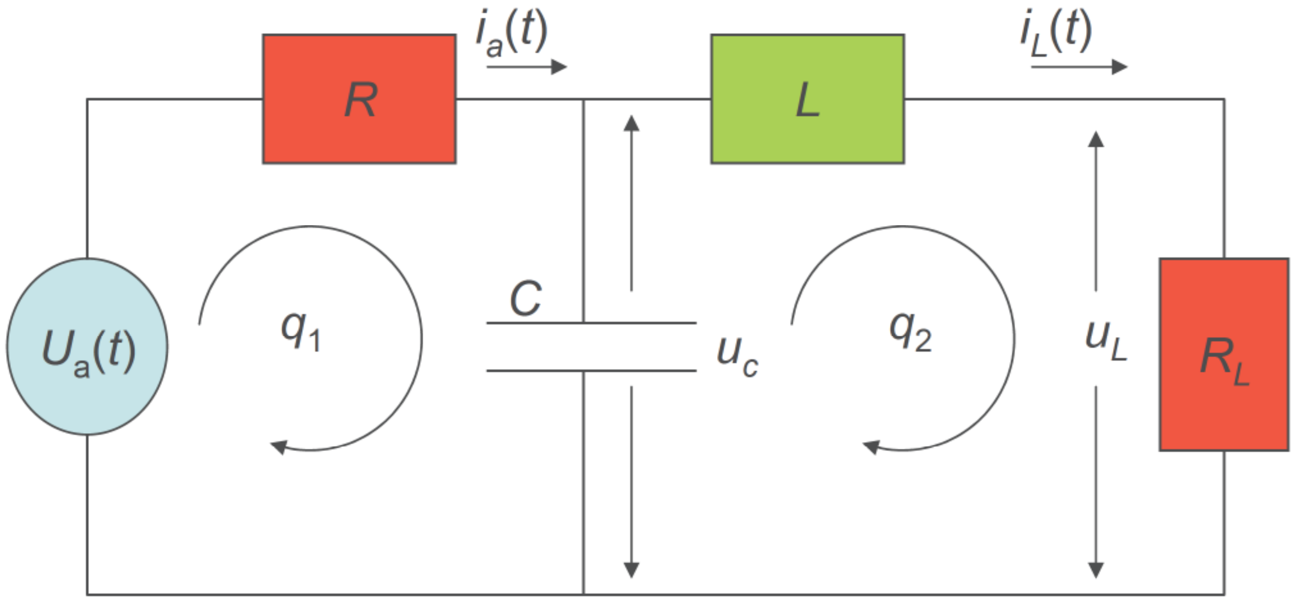
The total heat energy dissipated : $P = \frac{1}{2}R_1\dot{q}_1^2 + \frac{1}{2}R_2\dot{q}_2^2; \ \frac{\partial P}{\partial \dot{q}_1} = R_1\dot{q}_1$ and $\frac{\partial P}{\partial \dot{q}_2} = R_2\dot{q}_2$

$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_1}\right) - \frac{\partial K_e}{\partial q_1} + \frac{\partial P}{\partial \dot{q}_1} + \frac{\partial V}{\partial q_1} = Q_1; \ \frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_2}\right) - \frac{\partial K_e}{\partial q_2} + \frac{\partial P}{\partial \dot{q}_2} + \frac{\partial V}{\partial q_2} = 0$$

$$(L_1 + L_{12})\ddot{q}_1 - L_{12}\ddot{q}_2 + R_1\dot{q}_1 + \frac{q_1}{C_1} = U_a; \ -L_{12}\ddot{q}_1 + (L_2 + L_{12})\ddot{q}_2 + R_2\dot{q}_2 + \frac{q_2}{C_2} = 0$$

$$\ddot{q}_1 = \frac{1}{(L_1 + L_{12})}\left(-\frac{q_1}{C_1} - R_1\dot{q}_1 + L_{12}\ddot{q}_2 + U_a\right); \ddot{q}_2 = \frac{1}{(L_2 + L_{12})}\left(L_{12}\ddot{q}_1 - \frac{q_2}{C_2} - R_2\dot{q}_2\right)$$

Another Example



Use q_1 and q_2 as the independent generalized coordinates :

$$i_a = \dot{q}_1; i_L = \dot{q}_2; u_a(t) = Q_1$$

$$K_e = \frac{1}{2} L\dot{q}_2^2; \quad \frac{\partial K_e}{\partial q_1} = 0; \quad \frac{\partial K_e}{\partial \dot{q}_1} = 0; \quad \frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_1}\right) = 0$$

$$\frac{\partial K_e}{\partial q_2} = 0; \quad \frac{\partial K_e}{\partial \dot{q}_2} = L\dot{q}_2; \quad \frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_2}\right) = L\ddot{q}_2$$

$$\text{The total potential energy is}: V = \frac{1}{2}\frac{(q_1 - q_2)^2}{C}$$

$$\frac{\partial V}{\partial q_1} = \frac{q_1 - q_2}{C} \quad \text{and} \quad \frac{\partial V}{\partial q_2} = \frac{-q_1 + q_2}{C}$$

$$\text{The total dissipated energy is}: P = \frac{1}{2}R\dot{q}_1^2 + \frac{1}{2}R_L\dot{q}_2^2$$

$$\frac{\partial P}{\partial \dot{q}_1} = R\dot{q}_1 \quad \text{and} \quad \frac{\partial P}{\partial \dot{q}_2} = R_L\dot{q}_2$$

$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_1}\right) - \frac{\partial K_e}{\partial q_1} + \frac{\partial P}{\partial \dot{q}_1} + \frac{\partial V}{\partial q_1} = Q_1; \; \frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_2}\right) - \frac{\partial K_e}{\partial q_2} + \frac{\partial P}{\partial \dot{q}_2} + \frac{\partial V}{\partial q_2} = 0$$

$$R\dot{q}_1 + \frac{q_1 - q_2}{C} = u_a; \quad L\ddot{q}_2 + R_L\dot{q}_2 + \frac{-q_1 + q_2}{C} = 0$$

$$\dot{q}_1 = \frac{1}{R}\left(\frac{-q_1 + q_2}{C} + u_a\right); \; \ddot{q}_2 = \frac{1}{L}\left(-R_L\dot{q}_2 + \frac{q_1 - q_2}{C}\right)$$

By using Kirchhoff's law, we get

$$\frac{du_c}{dt} = \frac{1}{C}\left(-\frac{u_c}{R} - i_L + \frac{u_a(t)}{R}\right); \; \frac{di_L}{dt} = \frac{1}{L}(u_c - R_L i_L)$$