CONTROL I

ELEN3016

Discrete-Time Systems Properties

(Lecture 23)

Overview

- First Things First!
- Transient response
- Stability
- Root locus
- Discrete-time to continuous-time conversion
  - Filters and Zero-Order Hold (ZOH)
- Tutorial Exercises & Homework

First Things First!

- Course Review Meeting – date & time?

Discrete Systems Properties

- Transient Response

Relationship between paths in the s-plane and the z-plane follows from

$$z = e^{sT} = e^{j(\sigma + j\omega)T} = e^{j\sigma T} e^{j\omega T}.$$

... consider different contours in the s-plane and their dual in the z-plane.

Discrete Systems Properties

- Transient Response cont'd
  - For $z = e^{sT}$ with $T > 0$ we have that

$$|z| = \left|e^{(\sigma + j\omega)T}\right| = \left|e^{\sigma T}\right| \underbrace{\left|e^{j\omega T}\right|}_{=1} = e^{\sigma T}$$

  - For $\sigma > 0$ we have that $e^{\sigma T} > 1$.
  - For $\sigma = 0$ we have that $e^{\sigma T} = 1$.
  - For $\sigma < 0$ we have that $0 < e^{\sigma T} < 1$.

Discrete Systems Properties

- Transient Response cont'd

  We conclude that

  - the LHP of the s-plane maps to the interior of the unit circle in the z-plane;
  - the $j\omega$-axis in the s-plane maps to the circumference of the unit circle in the z-plane;
  - the RHP of the s-plane maps to the exterior of the unit circle in the z-plane.

Discrete Systems Properties

- Stability

  Consider the characteristic equation

  $$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$$

  Two methods exist for stability analysis:

  – Jury test (consult Burns),

  – Routh-Hurwitz test.

Discrete Systems Properties

- Stability – Routh-Hurwitz Test
  - The substitution $z = \frac{w-1}{w+1}$ (bilinear transformation maps the interior (exterior resp.) of the unit circle onto the LHP (RHP resp.) of the w-plane. The Char. Eq. becomes

$$a_n\left(\frac{w-1}{w+1}\right)^n + a_n\left(\frac{w-1}{w+1}\right)^{n-1} + \ldots + a_1\left(\frac{w-1}{w+1}\right) + a_0 = 0$$

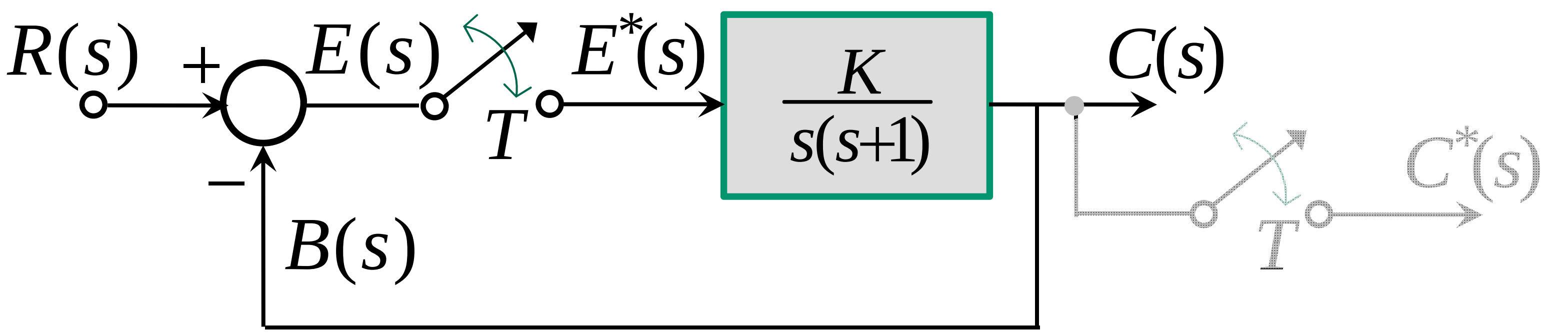
$$\Rightarrow a_n\left(w-1\right)^n + a_n\left(w-1\right)^{n-1}(w+1) + \ldots + a_1(w-1)\left(w+1\right)^{n-1} + a_0 = 0$$

Discrete Systems Properties

- Root Locus
  - The root locus rules, as for continuous systems, applies to the discrete Char. Eq.
  - The only difference is the interpretation of segments of loci that correspond to stable response.

Discrete Systems Properties

- Root Locus – Example



  – The closed-loop pulse transfer function is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)} \quad \text{with} \quad G(s) = \frac{K}{s(s+1)} \; .$$

Discrete Systems Properties

- Root Locus – Example cont'd

We proceed to calculate $G(z)$.

Using PFE, namely

$$G(s) = \frac{K}{s(s+1)} = K\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

we obtain

$$G(z) = \mathfrak{Z}[G(s)] = K\mathfrak{Z}\left(\frac{1}{s} - \frac{1}{s+1}\right) = K\left(\mathfrak{Z}\left(\frac{1}{s}\right) - \mathfrak{Z}\left(\frac{1}{s+1}\right)\right) =$$

$$= K\left(\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right) = K\frac{z\left(1-e^{T}\right)}{(z-1)\left(z-e^{-T}\right)}$$

Discrete Systems Properties

- Root Locus – Example cont'd

Char. Eq. of the closed-loop system is

$$1 + G(z) = 0$$

$$(z-1)\left(z-e^{-T}\right) + K\left(1-e^{-T}\right)z = 0$$

Open-loop zeros: $z = 0$

Open-loop poles: $z = 1, \quad z = e^{-T}$

Real axis segments implies break-in/away pts.

Discrete Systems Properties

- Root Locus – Example cont'd

Break-in & breakaway points:

$$-K\left(1-e^{-T}\right)=\frac{(z-1)\left(z-e^{-T}\right)}{z}$$

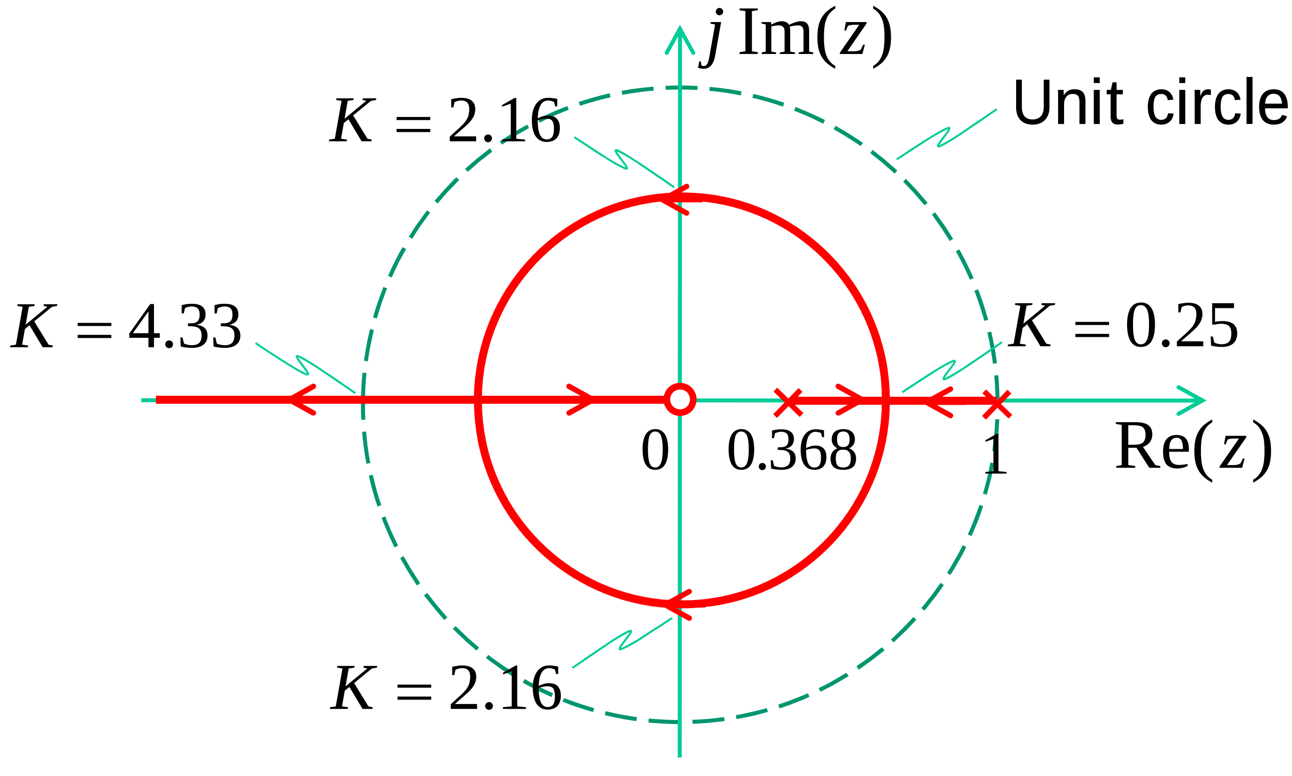
$$-\frac{d}{dz}\left(\left(1-e^{-T}\right)K\right)=\frac{d}{dz}\left(\frac{(z-1)\left(z-e^{-T}\right)}{z}\right)=\frac{z^2-e^{-T}}{z^2}=0$$

$$\Rightarrow \quad z=\pm e^{-\frac{T}{2}}$$

Discrete Systems Properties

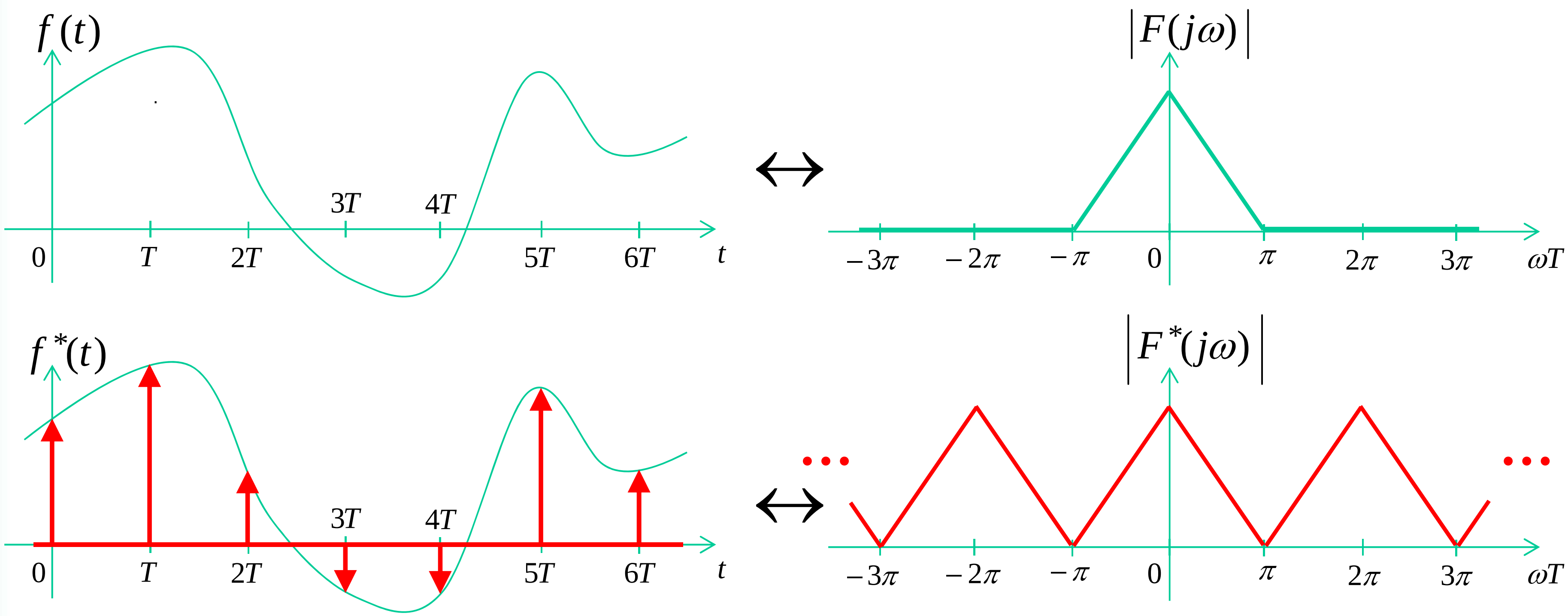
- Root Locus – Example cont'd

For $T = 1$ the root locus is shown below.



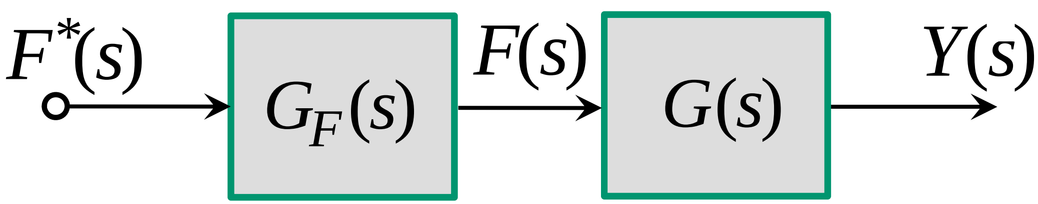
Discrete Systems Properties

- Discrete-Time to Continuous-Time – Filters

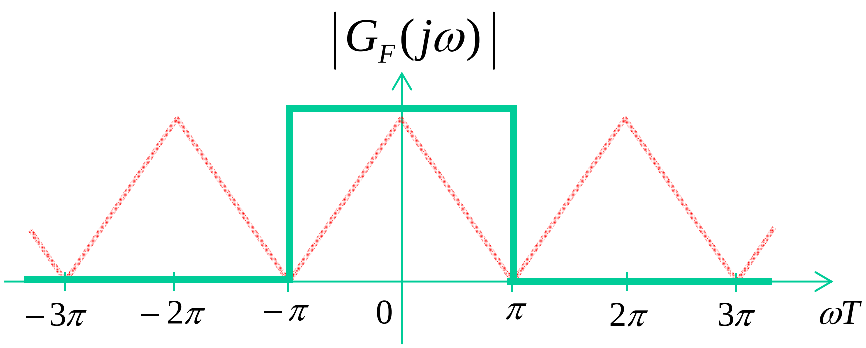


Discrete Systems Properties

- Discrete-Time to Continuous-Time – Filters

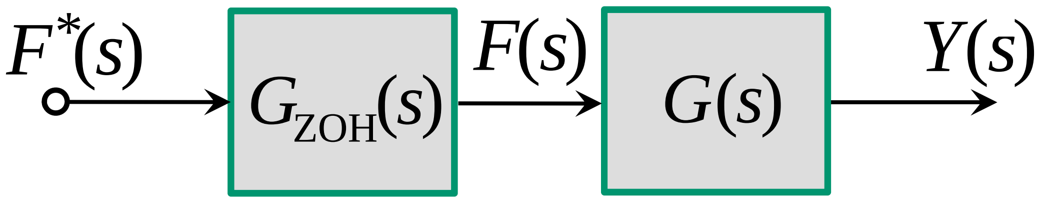


$$F(s) = G_F(s)F^*(s)$$

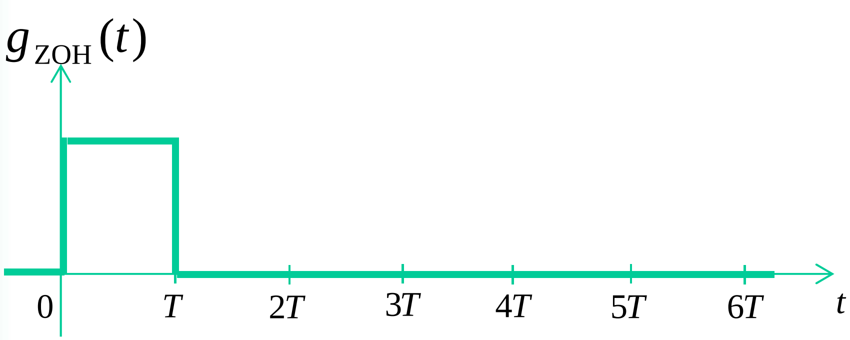


Discrete Systems Properties

- Discrete-Time to Continuous-Time – ZOH



$$G_{ZOH}(s) = \frac{1-e^{-sT}}{s}$$



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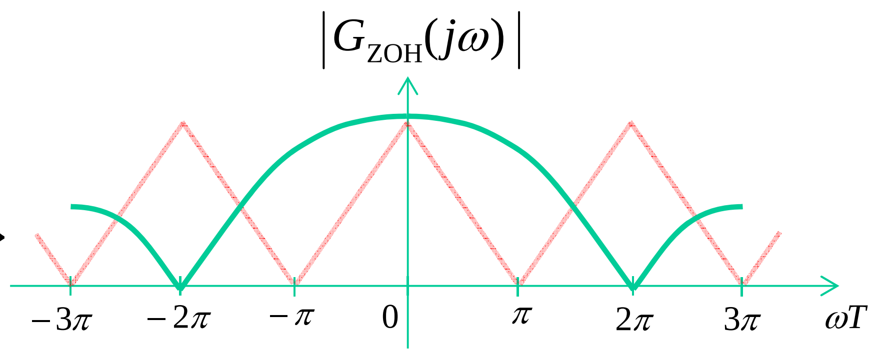
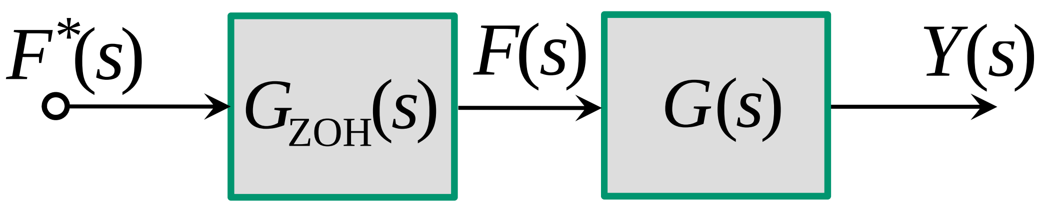
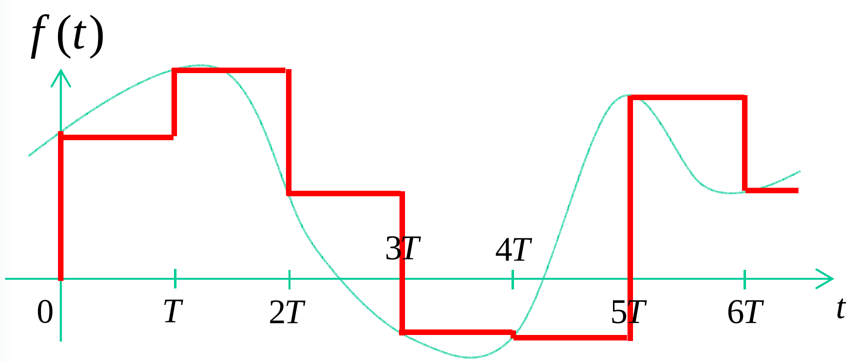


Discrete Systems Properties

- Discrete-Time to Continuous-Time – ZOH



$$G_{\mathrm{ZOH}}(s) = \frac{1-e^{-sT}}{s}$$



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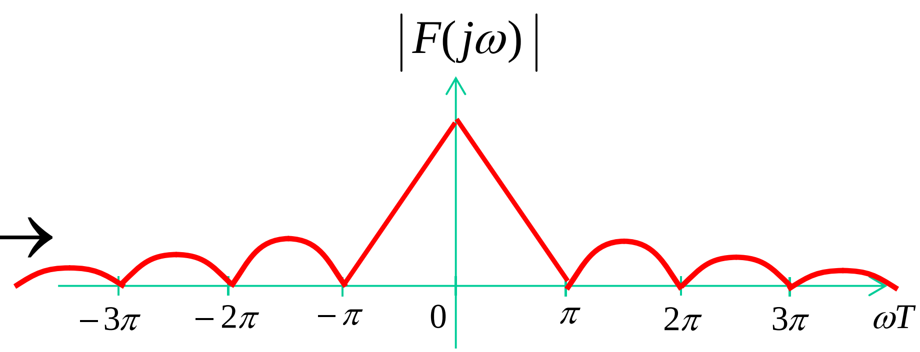

Tutorial Exercises & Homework

- Tutorial Exercises
  - Derive the frequency response of a ZOH with a sampling period of T s.
- Homework
  - Study all relevant sections in Burns.

Conclusion

- Transient Response
- Stability
- Root locus
- Discrete-time to continuous-time conversion
  - Filters and Zero-Order Hold (ZOH)
- Tutorial Exercises & Homework

Next Attraction! – Miss It & You'll Miss Out!

- Digital Control System Design (Burns, Chapter 7)

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**Thank you
for your interest!**