CONTROL I

ELEN3016

Stability of Dynamical Systems

(Lecture 11)

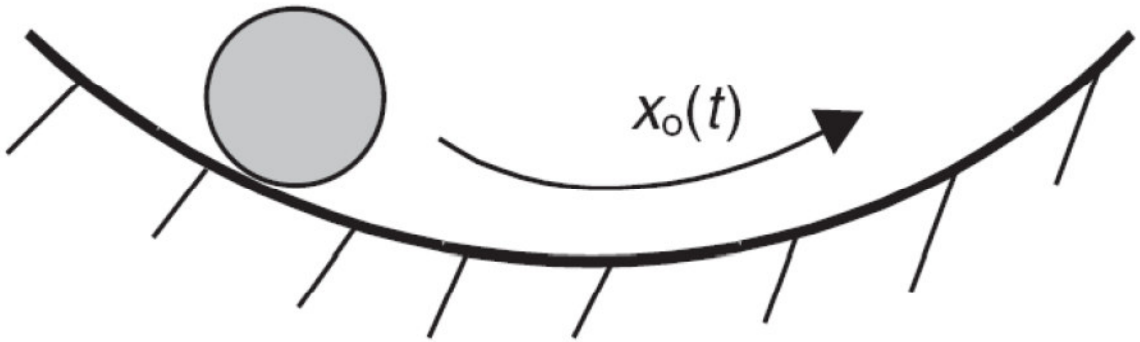
Overview

- First Things First!
- Introductory Examples
- Review of 2^{nd}-Order Systems
- Routh-Hurwitz Stability Criterion
- Tutorial Exercises & Homework

- Next Attraction!

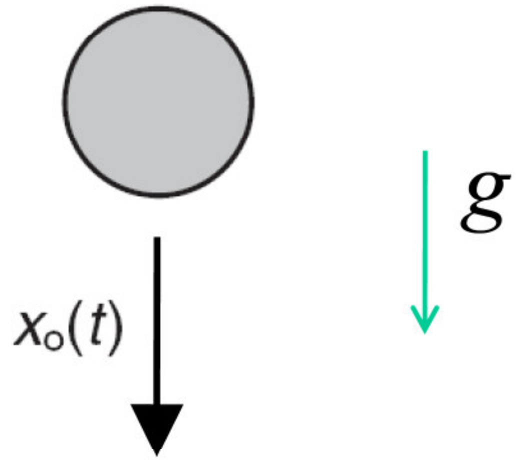
Introductory Example 1

- *Stable* time response



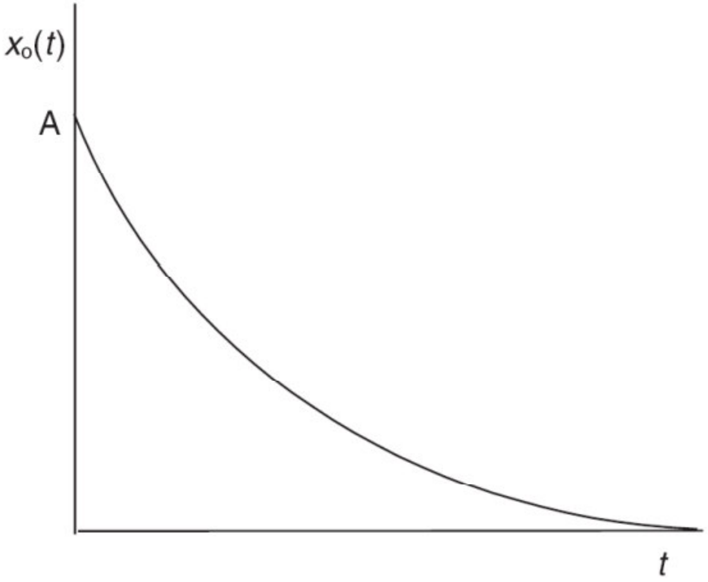
Introductory Example 2

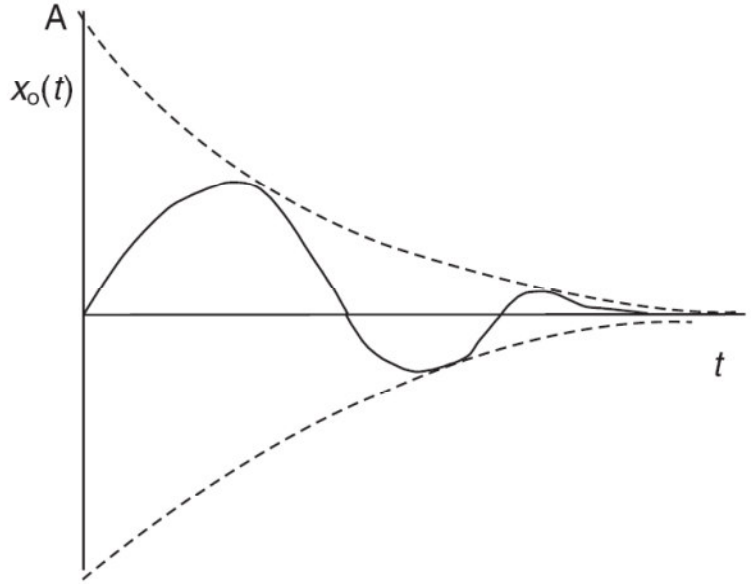
- Example of *unstable* time response



Types of Stable Time Response

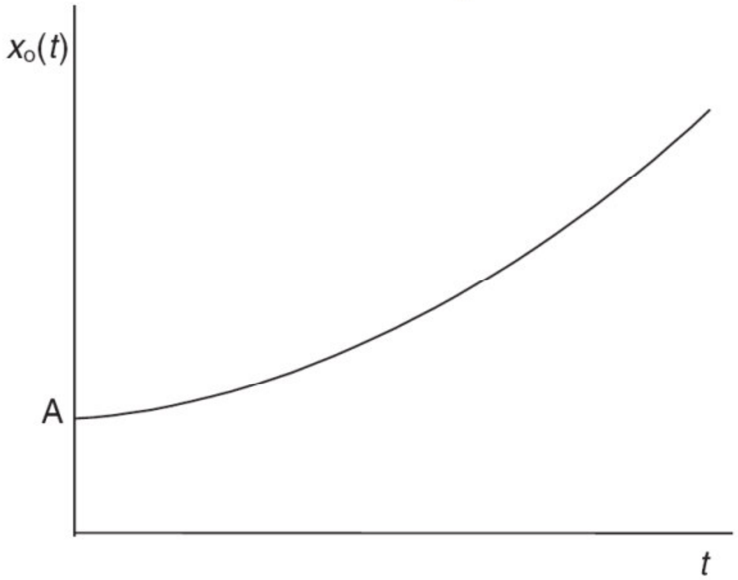
- Stable response: Non-oscillatory vs Oscillatory

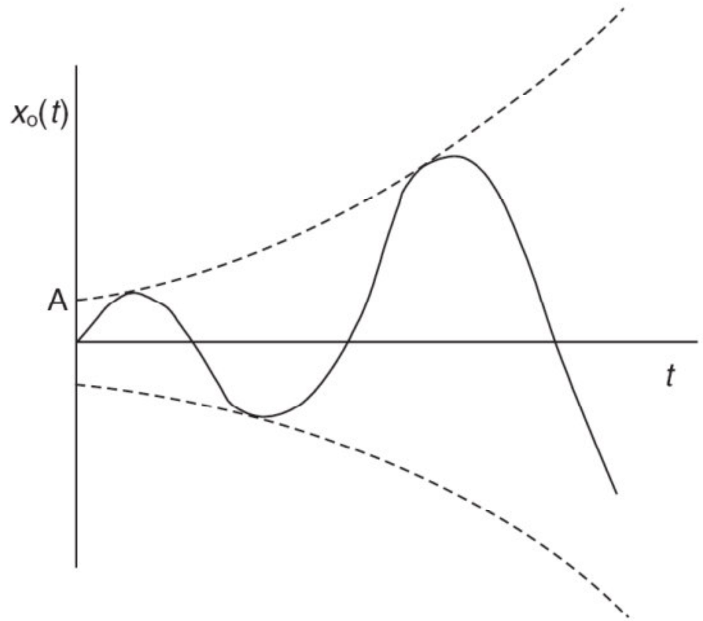




Types of Unstable Time Response

- Unstable response: Non-oscillatory vs Oscillatory





Characteristic Equation & Roots

- Characteristic equation of a 2nd-order system

$$as^2 + bs + c = 0 \qquad (5.5)$$

- Roots

$$s_1,\ s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (5.6)$$

$$\underbrace{a, b, c > 0}_{\text{That is, all of the same sign!}} \quad \Rightarrow \quad \begin{cases} \mathrm{Re}(s_1), \mathrm{Re}(s_2) < 0 \\ b > \sqrt{b^2 - 4ac} \end{cases}$$

Characteristic Equation & Roots

- Overdamped 2nd-order system $(b^2 - 4ac > 0)$

$$s_1 = -\sigma_1 \qquad s_2 = -\sigma_2 \tag{5.7}$$

- Critically damped 2nd-order system $(b^2 - 4ac = 0)$

$$s_1 = s_2 = -\sigma \tag{5.8}$$

- Underdamped 2nd-order system $(b^2 - 4ac < 0)$

$$s_1,\, s_2 = -\sigma \pm \mathrm{j}\omega \tag{5.9}$$

Characteristic Equation & Roots

- 2^{nd}-order system with $a, c > 0,\ \ b < 0$ implies
  a. $-4ac < 0 \Rightarrow\ b^2 - 4ac < b^2$ implying either
     i. complex conjugate pole pair if $b^2 - 4ac < 0$ **<u>or</u>**
     ii. two real poles of the same sign if $0 < b^2 - 4ac$.
  b. $-\frac{b}{2a} > 0$ i.e. at least one unstable pole.

Combining a.) & b.) we conclude that both poles (real or complex) are in the RHP giving an <u>unstable</u> system.

Characteristic Equation & Roots

- 2nd-order system with $b, c > 0, \;\; a < 0$ implies

  a. $-4ac > 0 \Rightarrow b^2 - 4ac > b^2$ from which we have two real poles of opposite signs since $0 < b < \sqrt{b^2 - 4ac}$.

  b. $-\frac{b}{2a} > 0$ i.e. at least one unstable pole.

Combining a.) & b.) ensures a single unstable pole in the RHP giving an <u>unstable</u> system.

Characteristic Equation & Roots

- 2^{nd}-order system with $a,b > 0,\ \ c < 0$ implies

  a. $-4ac > 0 \Rightarrow b^2 - 4ac > b^2$ from which we have two real poles of opposite signs since $0 < b < \sqrt{b^2 - 4ac}$.

  b. $-\dfrac{b}{2a} < 0$ i.e. at least one stable pole.

Combining a.) & b.) ensures a single unstable pole in the RHP giving an <u>unstable</u> system.

Stability of a LTI System

- Characteristic equation of an n^{th}-order system

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

- Necessary but not sufficient condition for a system to be stable:
  - The coefficients of the characteristic equation must all have the same sign and that none are zero.

Routh-Hurwitz Criterion

- Characteristic equation of an *n* th-order system

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \quad (5.11)$$

- Necessary and sufficient condition for a system to be stable:
  - All the *Hurwitz determinants* of the characteristic polynomial are positive or, equivalently,
  - All coefficients in the first column of the *Routh array* have the same sign.

Routh-Hurwitz Criterion

- Routh array

s^0	p_1			
s^1	q_1			
$\vdots$	$\vdots$			
s^{n-3}	c_1	c_2	c_3	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$

Routh-Hurwitz Criterion

- Routh array

s^0	p_1			
s^1	q_1			
$\vdots$	$\vdots$			
s^{n-3}	c_1	c_2	c_3	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$

Routh-Hurwitz Criterion

- Routh array

s^0	p_1			
s^1	q_1			
$\vdots$	$\vdots$			
s^{n-3}	c_1	c_2	c_3	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$

Routh-Hurwitz Criterion

- Expanding the Routh array's third row:

$$b_1 = \frac{1}{a_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix}$$

$$b_2 = \frac{1}{a_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ a_n & a_{n-4} \end{vmatrix}$$

$$\text{etc.}$$

  - Continue until the first zero appears. Then move to next row.

Routh-Hurwitz Criterion

- Routh array

$$b_1 = \frac{1}{a_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix}$$

s^0	p_1			
s^1	q_1			
$\vdots$	$\vdots$			
s^{n-3}	c_1	c_2	c_3	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$

Routh-Hurwitz Criterion

- Routh array

$$b_2 = \frac{1}{a_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ a_n & a_{n-4} \end{vmatrix}$$

s^0	p_1			
s^1	q_1			
$\vdots$	$\vdots$			
s^{n-3}	c_1	c_2	c_3	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$

Routh-Hurwitz Criterion

- Expanding the Routh array's fourth row:

$$c_1 = \frac{1}{b_1}\begin{vmatrix} b_1 & b_2 \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$c_2 = \frac{1}{b_1}\begin{vmatrix} b_1 & b_3 \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

etc.

  – Continue until the first zero appears. Then move to next row. Terminate with a complete row of zeros.

Routh-Hurwitz Criterion

- Routh array

$$c_1 = \frac{1}{b_1}\begin{vmatrix} b_1 & b_2 \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

s^0	p_1			
s^1	q_1			
$\vdots$	$\vdots$			
s^{n-3}	c_1	c_2	c_3	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$

Routh-Hurwitz Criterion

- Routh array

$$c_2 = \frac{1}{b_1} \begin{vmatrix} b_1 & b_3 \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

s^0	p_1			
s^1	q_1			
$\vdots$	$\vdots$			
s^{n-3}	c_1	c_2	c_3	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$

Routh-Hurwitz Criterion

- Example 5.1

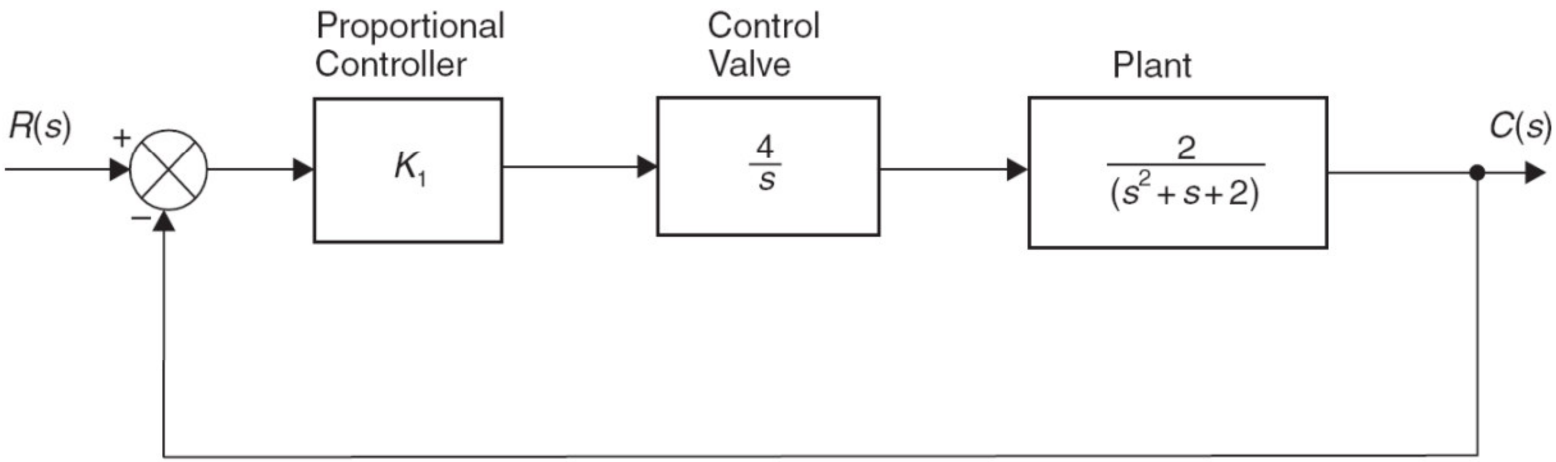
$$s^4 + 2s^3 + s^2 + 4s + 2 = 0$$

Routh array:

$$\begin{array}{c|ccc} s^0 & 2 & & \\ s^1 & 8 & & \\ s^2 & -1 & 2 & \\ s^3 & 2 & 4 & \\ s^4 & 1 & 1 & 2 \\ \hline \end{array}$$

Routh-Hurwitz Criterion

- Example 5.2



Find the minimum proportional gain for which the system is "only just unstable" (marginally stable).

Routh-Hurwitz Criterion

- Example 5.2 (continued)

Characteristic equation: $(K = 8K_1)$

$$1 + G(s)H(s) = 1 + \frac{K}{s(s^2 + s + 2)} = 0$$

$$s^3 + s^2 + 2s + K = 0 \qquad (5.30)$$

Routh-Hurwitz Criterion

- Example 5.2 (continued)

Routh array:

$$\begin{array}{c|cc} s^0 & K & \\ s^1 & (2-K) & \\ s^2 & 1 & K \\ s^3 & 1 & 2 \end{array}$$

Routh-Hurwitz Criterion

- Example 5.2 (continued)

Routh array:

$$
\begin{array}{c|cc}
s^0 & K & \\
s^1 & (2-K) & \\
s^2 & 1 & K \\
s^3 & 1 & 2 \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{l}
K \geq 2 \\
K_1 = 0.25
\end{array}
$$

Routh-Hurwitz Criterion

- Example 5.2 (cont'd)

  Study the remaining part of Example 5.2 as well as Section 5.2.2 on special cases of the Routh array.

Tutorial Exercises & Homework

- Tutorial Exercises
  - Burns, Examples 5.12 and 5.13
- Homework
  - Burns, Example 5.2 and Sec. 5.2.2

Conclusion

- First Things First!
- Introductory Examples
- Review of 2nd-Order Systems' Stability
- Routh-Hurwitz Stability Criterion
- Burns, Sec 5.2.2 (Self-study!)
- Tutorial Exercises & Homework

Next Attraction! – Miss It & You'll Miss Out!

- The Root Locus Technique (Burns, Chapter 5)

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Thank you!

Any Questions?